On the Transformation of Integrals.

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§ 1. It is known that, if

$$x = x(u, v), \qquad y = y(u, v),$$

be one-valued continuous functions of (u, v), possessing continuous differential coefficients, while $f(x, y)$ is any continuous function of (x, y),

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d f\{x(u, v), y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv, \qquad \text{(I)}$$

where the integration on the left-hand side is taken over the area of the plane curve, C, which is the image in the (x, y)-plane of the rectangle $(a, b;\ c, d)$ in the (u, v)-plane. Here it is tacitly assumed that C divides the plane into two distinct parts, a limitation which, however, disappears when we employ the definition of integration over an area which I have found it necessary to introduce into analysis.*

§ 2. If we denote by $F(x, y)$ one of the indefinite integrals of $f(x, y)$ with respect to x, and by

$$x = x(t), \qquad y = y(t),$$

suitably chosen equations of the curve C, the surface integral on the left-hand side of (I) may be replaced by a contour integral (*loc. cit.*). In fact, we may then write (I) in the form

$$\int_{t_0}^{t_1} F\{x(t), y(t)\}\, dy(t) = \int_a^c \int_b^d f\{x(u, v), y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv, \qquad \text{(II)}$$

an identity which holds good however the curve C cuts itself, provided only the function $y(t)$ has bounded variation, or, as I have elsewhere† phrased it, provided C is a semi-rectifiable curve.

§ 3. The whole of the theory of the transformation of the variables in multiple integrals may be said to depend on identities of the forms (I) and (II). The problem of determining the most general conditions under which they hold good must be regarded as one of the most fundamental in the whole range of analysis.

* See a recent paper by the author "On Integration over the Area of a Curve and Transformation of the Variables in a Multiple Integral," 1920. To appear in the 'Proc. L. M. S.'

† Address to the Congrès de Mathématiciens at Strasbourg, 1920.

Employing a method explained in my paper "On a Formula for an Area,"* I recently obtained (*loc. cit. prim.*) such a set of conditions. Still more recently,† I have constructed a new method of dealing with these matters, and I propose in the present communication to obtain by means of it a second, perhaps still more strikingly, general set of sufficient conditions, involving the notion of *associated summabilities*, which I had occasion some years back to explain to the Society.‡ This method seems of itself sufficiently remarkable to merit attention, not only for its own sake, but also for the numerous applications of which it appears to be capable.

The set of conditions (III) obtained is as follows:—

(i) $x(u, v)$ *and* $y(u, v)$ *are each absolutely continuous functions of each of the variables* u *and* v *separately*:—

(ii) *their partial derivates* $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, *and* $\partial y/\partial v$ *have absolutely convergent Lebesque double integrals, and either*,§—

(iiia) *in each of the pairs of functions* $(\partial x/\partial u, \partial y/\partial v)$, $(\partial x/\partial v, \partial y/\partial u)$, *one function is bounded, or else*—

(iiib) *the functions in each pair have associated summabilities.*

§ 4. It is of importance to remark that the theorem just stated does not cease to hold when $f(x, y)$ is no longer continuous. By employing the method of sequences, we can successively replace $f(x, y)$ by functions of increasing degree of generality, and obtain, for example, the theorem that the integral of any bounded function $f(x, y)$ over a generalised area is expressible in the form (I). For the details of the argument we refer to the paper first cited above.|| It is not necessary to repeat it here, as the reasoning does not depend on the particular form of the conditions hypothecated for the functions $x(u, v)$ and $y(u, v)$.

In what follows, we shall accordingly suppose $f(x, y)$ continuous.

§ 6. We begin by supposing that $y(u, v)$ possesses the property of having

* 'Proc. L. M. S.,' Ser. 2, vol. 18, pp. 239–374.

† "A New Method in the Theory of Areas," 1920. To appear in the 'Proc. L. M. S.'

‡ "On Classes of Summable Functions and their Fourier Series," 1912, 'Roy. Soc. Proc.,' A, vol. 87.

§ Conditions (iii*a*) and (iii*b*) refer to each pair separately, thus, for instance (iii*a*), may hold for $\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial v}\right)$ and (iii*b*) for $\left(\frac{dx}{\partial v}, \frac{\partial y}{\partial u}\right)$. Moreover, in different intervals these alternatives may vary, and the summabilities may also vary, provided they remain associated.

|| See the paper on "A New Method . . ." (*loc. cit.*), where $\bar{Q}(u, v)$ is used for what is here designated $Q(u, v)$.

partial differential coefficients which are continuous functions of (u, v), while $x(u, v)$ has only the properties stated in III. Writing

$$Q(u, v) = \frac{1}{hk}\int_u^{u+h}\int_v^{v+k} x(u, v)\, du\, dv = \int_0^1\int_0^1 x(u+t_1h, v+t_2k)\, dt_1\, dt_2,$$

and therefore

$$\frac{\partial Q}{\partial u} = \frac{1}{hk}\int_u^{u+h}\int_v^{v+k} \frac{\partial x}{\partial u}\, du\, dv = \int_0^1\int_0^1 \frac{\partial}{\partial u} x(u+t_1h, v+t_2k)\, dt_1\, dt_2,$$

we see that $Q(u, v)$ has precisely the same properties as those we are retaining for $y(u, v)$. Thus (II) holds, if on the right-hand side we replace $x(u, v)$ by $Q(u, v)$, and at the same time make the corresponding change of $x(t)$ to $Q(t)$ on the left-hand side.

Now, if $x(u, v)$ is a continuous function of (u, v), we can determine constants h_0 and k_0, so that for *all* values of $|h| \leq h_0$ and of $|k| \leq k_0$,

$$|x(u+t_1h, v+t_2h)-x(u, v)| \leq \epsilon,$$

t_1 and t_2 lying between 0 and 1, whence also

$$|Q(u, v)-x(u, v)| \leq \epsilon.$$

In this case, $x(u, v)$ is the *unique double limit* of $Q(u, v)$ when $(h, k) \rightarrow (0, 0)$.

If, however, as in our conditions, $x(u, v)$ is only assumed to have the property of continuity with respect to each variable u and v separately, we obtain similarly $x(u, v)$ as the *repeated* limit of $Q(u, v)$, when either $h \rightarrow 0$, then $k \rightarrow 0$, or when first $k \rightarrow 0$, then $h \rightarrow 0$. Also, whereas in the former case, the approach to the double limit was seen to be uniform doubly, now it is uniform with respect to each variable in turn,* and we may still assume that, for all values under consideration,

$$|Q(u, v)-x(u, v)| \leq \epsilon,$$

as h and k approach their limits in the given order.

§ 7. Taking then the equations we have obtained

$$\int_{t_0}^{t_1} F\{Q(t), y(t)\}\, dy(t) = \int_a^c\int_b^d f\{Q(u, v), y(u, v)\}\, \frac{\partial(Q, y)}{\partial(u, v)}\, du\, dv, \qquad (1)$$

where

$$Q(t) = Q(u, v) = \int_0^1\int_0^1 x(u+t_1h, v+t_2k)\, dt_1\, dt_2, \qquad (2)$$

* For instance, taking $h \rightarrow 0$, $k \rightarrow 0$, we have for any chosen value of k a value $h_{0,k}$ depending on t_2k, but independent of h or t_1, such that

$$|x(u+t_1h, v+t_2k)-x(u, v+t_2k)| < \tfrac{1}{2}\epsilon.$$

Also we can choose our value of k to be less than a certain k_0, for which

$$|x(u, v+t_2k)-x(u, v)| < \tfrac{1}{2}\epsilon,$$

and, therefore,
$$|x(u+t_1h_1)v+t_2k)-x(u, v)| < \epsilon,$$

and
$$|Q(u, v)-x(u, v)| < \epsilon.$$

for values of (u, v) on the perimeter of the rectangle $(a, b;\ c, d)$, and

$$|\mathrm{Q}(u, v)-x(u, v)| \leq \epsilon. \tag{3}$$

We have now to carry out and to justify the procession to the limit (repeated limit, $h \rightarrow 0$, $k \rightarrow 0$ or $k \rightarrow 0$, $h \rightarrow 0$) under the functional and integral signs, so as to obtain the equation (II).

§ 8. We commence with the left-hand side of (1).

We have, since $\mathrm{F}(x, y)$ is continuous with respect to (x, y),

$$|\mathrm{F}\{\overline{\mathrm{Q}}(t), y(t)\}-\mathrm{F}\{x(t), y(t)\}| \leq e, \tag{4}$$

where e is any chosen quantity, provided, in virtue of (3), ϵ be chosen conveniently small. Therefore the left-hand side of (4), as we proceed to the limit considered, tends uniformly to zero, and, therefore, we may integrate term-by-term with respect to the function of bounded variation $y(t)$.

Hence the left-hand side of (1) tends to the required form, namely, the left-hand side of (II).

§ 9. We now proceed to discuss the right-hand side of (1). We break up the discussion into two parts, taking each term of the Jacobian separately, and proceed first to show that

$$\int_a^c \int_b^d \left\{f(\mathrm{Q}, y) \frac{\partial \mathrm{Q}}{\partial u} \cdot \frac{\partial y}{\partial v} du\, dv - f(x, y) \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v}\right\} du\, dv \rightarrow 0. \tag{5}$$

We may write the integrand here as the sum of two terms:—

$$f(x, y) \frac{\partial y}{\partial v}\left\{\frac{\partial \mathrm{Q}}{\partial u}-\frac{\partial x}{\partial u}\right\}+\frac{\partial \mathrm{Q}}{\partial u} \cdot \frac{\partial y}{\partial v}\{f(\mathrm{Q}, y)-f(x, y\}. \tag{α}$$

Now I have already proved in "A New Method in the Theory of Areas" that the double integral of the first of these two terms tends uniformly to zero, if the factor $f(x, y)$ be omitted. The proof, which only assumes that x and y obey the conditions (III), is not affected by the multiplication of the function $\partial y / \partial v$ by the continuous function $f\{x(u, v),\ y(,u\ v)\}$. The new function is, in fact, still independent of h and k, and has the same summability properties as $\partial y / \partial v$. Thus we may assume the proof, and write,

$$\int_a^c \int_b^d \left\{\frac{\partial \mathrm{Q}}{\partial u}-\frac{\partial x}{\partial u}\right\} f(x, y) \frac{\partial y}{\partial v} du\, dv \rightarrow 0. \tag{6}$$

The second term in (α) is the product of $\partial \mathrm{Q} / \partial u \,.\, \partial y / \partial v$ into a factor $\{f(\mathrm{Q}, y)-f(x, y)\}$ which, in virtue of (3), is numerically less than e,—since $f(x, y)$ is continuous with respect to (x, y),—provided ϵ be chosen conveniently small. When integrated, therefore, the result is numerically

$$< e \int_a^c du \int_b^d dv \int_0^1 \int_0^1 \left|\frac{\partial y}{\partial v}\right| \left|\frac{\partial}{\partial u} x(u+t_1 h,\ v+t_2 k)\right| dt_1\, dt_2,$$

$$< e \int_0^1 dt_1 \int_0^1 dt_2 \int_a^c \int_b^d \left|\frac{\partial y}{\partial v}\right| \left|\frac{\partial}{\partial u} x(u+t_1 h_1,\ v+t_2 k)\right| du\, dv,$$

where we are justified in changing the order of integration, since the quadruple integral exists by reason of the association of the summabilities of $\partial y/\partial v$ and $\partial x/\partial u$.* Moreover, by the fundamental† theorem in the theory of such pairs of functions, the inside double integral is less than a finite constant B for all the values of (h, k) considered. Hence, since e is as small as we please,

$$\int_a^c \int_b^d \frac{\partial Q}{\partial u} \cdot \frac{\partial y}{\partial v} \{f(Q, y) - f(x, y)\}\, du\, dv \to 0. \tag{7}$$

By (6) and (7) the double integral of (α) tends to zero, which proves (5).

Similarly the second term of the Jacobian may be treated, whence, combining the two terms, the required result follows, namely, the right-hand side of (1) tends to that of (II).

We repeat once more that the limit we have here taken is, with the conditions imposed on $x(u, v)$ in (III), the repeated limit, when first $h \to 0$, then $k \to 0$, or when first $k \to 0$, then $h \to 0$.

§ 10. We have already remarked that the theorem borrowed from "A New Method in the Theory of Areas" holds good when neither x nor y have continuous differential coefficients, but the conditions (III) hold good. We can therefore repeat the argument, and deduce the truth of (II) for the general case from that just proved, in which one of the pair (x, y) has continuous differential coefficients. All that is necessary is to make a certain change in dealing with the left-hand side of (II).

For this purpose, we remark that the left-hand side of (II) may be written

$$-\int G\{x(t), y(t)\}\, dx(t),$$

where $G(x, y)$ is an indefinite integral with respect to y of $f(x, y)$. Here the function with respect to which we integrate satisfies conditions (III), while the other function $y(u, v)$, involved implicitly in the integrand, has continuous differential coefficients. Thus, replacing $y(u, v)$ by

$$R(u, v) = \frac{1}{hk} \int_a^c \int_b^d y(u, v)\, du\, dv,$$

and carrying out precisely the same argument as in § 9, the proof of our theorem is completed.

* *Cp.* "A New Method in the Theory of Areas."

† *Ibid.*, proof of Lemma 3. Theorem originally given in "On Classes of Summable Functions . . .," *loc. cit.*, p. 226.